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An aerial photograph of a vast, heavily crevassed ice field, likely in the Arctic. The ice is characterized by a complex network of dark, winding cracks and ridges, creating a textured, almost cellular appearance. The lighting is bright, highlighting the ridges and casting deep shadows in the crevasses. The overall tone is monochromatic, with various shades of gray and white.

AIDJEX BULLETIN

ARCTIC
ICE
DYNAMICS
JOINT
EXPERIMENT

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Theoretical Discussions

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Arctic Ice Dynamics Joint Experiment
Joseph O. Fletcher, Program Coordinator
Division of Marine Resources
University of Washington
Seattle, Washington 98105

FOREWORD

Much progress in formulating theory of pack-ice dynamics is needed to make the AIDJEX field measurements most rewarding. Unresolved questions remain which affect experiment design. For example, what are the most appropriate space scales for various stress and strain measurements; how large are the sampling errors; and what are realistic accuracy requirements for field measurements? Also, mathematical simulation models must be further developed both to fully utilize AIDJEX results for ice prediction and to complement global models of atmospheric and oceanic circulation.

A working group on Ice Dynamics Analysis and Simulation will consider these and other theoretical questions at a seminar on November 13 and 14 at the University of Washington.

The informal papers in this Bulletin demonstrate the continuing AIDJEX discussion that has been going on here. They are background material for the working group as well as an opportunity for all AIDJEX participants to listen in on the continuing discussion.

The next issue will extend the theoretical discussion to include Soviet authors in an attempt to more clearly reveal the frontier of understanding.

Joseph O. Fletcher
AIDJEX Coordinator

INTRODUCTION

THE KINEMATICS AND MECHANICAL BEHAVIOR OF PACK ICE: THE STATE OF THE SUBJECT

by

D. A. Rothrock

How does pack ice move? We know that it is pushed about on the surface of the oceans by the winds and the ocean currents, by body forces, and by the jostling of neighboring ice, but our quantitative understanding of both the driving forces and the mechanical behavior of pack ice is meager.

The body forces are the best understood but the least important. The stress caused by the time- and space-dependent turbulent flows in the fluids bounding the ice above and beneath can be crudely predicted. The least understood of the elements of pack-ice dynamics is the mechanical behavior of the pack. And in order to construct theories or models of this behavior, one must first have an understanding of the kinematics of the pack ice. This subject, too, is poorly understood. It is with these topics--the kinematics and the mechanical behavior of pack ice--that this issue of the AIDJEX Bulletin is concerned.

PACK-ICE KINEMATICS

The velocity field of pack ice varies in space and in time, so that even a large-scale feature of the motion, such as the Beaufort Sea Gyre, is continuously changing in size, shape, intensity, and location.

Our experimental data about this complicated velocity field consist of tracks of single floes and tracks of arrays of several floes. Only a few arrays have been tracked and, except for these, data have seldom been obtained for more than one or two tracks at any given time. Hence, we know almost nothing of the time-dependence of the field of motion.

If the data from many tracks are superimposed, a general pattern emerges which delineates the Beaufort Sea Gyre and the Transpolar Drift Stream. But it must be realized that such a pattern may bear scant relation to an actual instantaneous velocity field.

The array data reported by Bushuyev et al. (1967) portray most realistically of any to date a real phenomenon on temporal and spatial scales relevant to ice movement in the central arctic. Locations of the four stations, lying roughly at the corners of a square 100 km on a side, were recorded daily. The spatial coherence of the motion is remarkable--a similar curlicue occurs nearly simultaneously in each of the otherwise fairly straight tracks.

Can the ice pack be regarded as a continuum? That is, if properties are resolved only to the nearest, say, 100 km, will they appear to be continuously varying? The Soviet data just mentioned seem to suggest so, but there are so few array data yet available that they are barely significant statistically. Thorndike (p. 51 of this Bulletin) has described a method for calculating the strain-rate tensor from array data on the assumption that a continuous deformation field exists.

Can pack ice behave as a noncontinuum? Suppose that, in a significant regime of motion, floes freeze together into aggregate plates which may measure hundreds of kilometers across, and which can retain their integrity for days or longer before fracturing and refreezing into new plates. The relative motion would necessarily be concentrated at the boundaries of the plates which would be regions of large shearing motion, of intense pressing, and of generally violent floe interaction and deformation. Could the Soviet array have been locked into such a plate or bisected by a slip line between plates? There is no way to tell. Data currently available are insufficient to test this possibility.

There is reasonable observational evidence that pack ice exhibits deformation mechanisms which operate on large scales. In some regions, especially along the boundaries of the Transpolar Drift Stream, systems of polynyas can be observed, regularly spaced, tens of kilometers apart.

We cannot at present say whether, on the time scale of a day, pack ice can be regarded as a continuum. Data from four-station arrays are inadequate to resolve this question. What is needed is simultaneous data sampled at enough points to define the flow field over most of the Arctic Ocean. Such a project is planned by Soviet scientists for the early 1970's.

PROPERTIES OF PACK ICE

It stands to reason that, on a large scale, the kinematics and mechanical behavior of pack ice strongly interact with properties which characterize pack ice--especially with the statistics of the floe geometries. To date, this subject has received almost no theoretical or experimental attention.

The prediction of large-scale mechanical behavior from assumptions about actual mechanisms of floe interactions might be called a "mechanistic" theory. Presumably, such a theory would require information about the relative abundance of different size floes, about shape of floes, and about the arrangement of floes--especially the amount of open water and, perhaps, its geometry. The first mechanistic theories are those by Timokhov (1967 b and c) and Solomon (p. 33 of this Bulletin).

Not only does the motion depend on pack-ice properties, but the motion also is one of several factors which determine and maintain the state of the pack--in particular, the topographical features of the upper and lower surfaces. These features are maintained by a dynamic balance between, on the one hand, the crushing and piling of ice into ridges or hummocky areas and, on the other, the leveling tendencies of preferential melting and freezing and of snowfall. The need for an understanding of the surface topography--especially for predicting wind and water stress, but also for estimating the volume and, thus, the mass of ice--has motivated a good deal of research on the structure of pressure ridges. This work has been recently reviewed by Weeks and Kovacs (1970). The kinematics of ridge formation are now partially understood, although

the dynamics of these processes and their relation to large-scale deformation has received no attention.

Ridging is just one aspect of floe interaction; it has attracted attention both because it is an easily visible feature and because of the importance of the topography it creates. But of other mechanisms of floe interactions--for example, simple bumping and pushing--little is known. Some work on this subject is planned for the 1971 AIDJEX pilot studies.

Soviet scientists have recently shown interest both in the formulation of mechanistic models and in the measurement of properties which may be relevant to these models. Gorbunov and Timokhov (1968) have described measurements of floe sizes, relative velocities and angular velocities by means of aerial photographs. Some of the correlations they find between various properties seem tenuous; and, unfortunately, there are no data to permit correlations between these properties and large-scale deformation, winds, and currents.

With the strides being made in remote sensing of ice and with the potential of automated pattern recognition, our understanding of the relationships between motion and pack properties should mature rapidly.

THE EQUATIONS OF MOTION

It is assumed in the remainder of this note and in the other notes in this Bulletin that field variables possess as many continuous derivatives as appear in the equations. These variables may correspond to actual pack variables only in certain spatial or temporal averages. We could later generalize this restriction to allow for lines of discontinuity within the field to describe such large-scale features as Peary's "Big Lead."

We apply the balance principles only to the ice as opposed to, say, some upper oceanic layer which includes water and ice. The concentration (or mass per unit area) we denote by m . An element of matter has velocity \bar{u} or (u_1, u_2) in a plane defined by Cartesian axes (x_1, x_2) .

The rate of increase of mass in the area with respect to time t must equal the rate of net influx of mass plus the rate of formation Φ of mass by accretion, by the freezing over of open water, and by solid precipitation deposited on existing ice. Thus, we have

$$\frac{\partial m}{\partial t} = - \operatorname{div} \cdot (m\bar{u}) + \Phi \quad . \quad (1)$$

Note that the process of ridging does not contribute to Φ but simply rearranges existing ice in a way which usually favors a local convergence of the velocity field.

The mass balance equation (1) is exact and independent of any assumptions about the existence of the nature of floes or of open water or of floe interactions. However, since open water is so important in controlling vertical heat exchange and facilitating navigation, it would be useful to introduce it into the theory. We thus replace the notion of a single material by a model with two species--open water and ice. The fraction of area covered by ice is called the compactness N , and must satisfy

$$0 \leq N \leq 1 \quad .$$

If the ice were spread evenly over the fraction of area it actually covers (and not over open water), its thickness would be H . (Were it spread evenly over the whole area, the thickness would be $H \cdot N$.)

The definition of compactness is fairly precise in theory. Because sea ice contains voids, there is some latitude in specifying the mean density ρ of ice and the thickness H --except that their definitions must be consistent and satisfy the identity

$$m = \rho H N \quad . \quad (2)$$

We take ρ to be a constant, whereas H and N can now both be independent variables.

The equation of mass balance can be replaced by two equations:

that of thickness balance

$$\rho N \frac{DH}{Dt} = \Phi_1 + m \Psi \quad , \quad (3)$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u} \cdot \nabla$

Φ_1 = the rate of formation of mass by accretion and precipitation

Ψ = the logarithmic rate of increase of thickness by deformation, as seen moving with the ice;

and the equation of compactness balance

$$\rho N \frac{DN}{Dt} = -m \operatorname{div} \cdot \bar{u} + \Phi_2 - m \Psi \quad , \quad (4)$$

where Φ_2 = the rate of formation of mass by freezing of open water.

Equation (4) is obtained by subtracting (3) from (1) with the proviso that all mass-forming processes are accounted for in either Φ_1 or Φ_2 so that

$$\Phi_1 + \Phi_2 = \Phi \quad .$$

The equation of conservation of mass per unit area

$$\frac{\partial m}{\partial t} = -\operatorname{div} \cdot (m\bar{u})$$

was first suggested to apply to pack ice by Drogaitsev (1956) (with a sign error). Nikiforov (1957) hypothesized that the same equation should be applied to compactness rather than to concentration. A mass balance equation equivalent to (1) was discussed by Untersteiner (1963). Timokhov (1967c) suggested an equation for compactness in which Ψ is included and Φ_2 is not. The most satisfactory inclusion of considerations of mass balance has been Doronin's (1970) calculation of summer ice conditions in the Kara Sea. In that work, the momentum equation, as well as equations (3) and (4), both with $\Psi = 0$, are solved simultaneously.

(Even though it does not appear in Doronin's differential equation of compactness balance, a term like Φ_2 is included in the calculation.)

Our understanding of the term Ψ is extremely meager, and it is not surprising that it has, to date, been included in only one investigation (Timokhov, 1967c) of ice dynamics. Since deformation mechanisms almost invariably pile ice up, Ψ is almost invariably zero or positive. The first new attempts to include it may treat it as a parameter. Eventually it must be included as a function of kinematics and pack-ice properties.

Unlike the equations of concentration, thickness, and compactness balance, the momentum equation

$$m \frac{Du_i}{Dt} = mB_i + S_i + \frac{\partial \sigma_{ij}}{\partial x_j}, \quad \text{for } i, j = 1, 2, \quad (5)$$

has received a good deal of attention.

The body force per unit mass B_i includes the Coriolis force and the component of gravity along the sea surface.

The net stress S_i applied at both surfaces includes the air and water stress and any addition of momentum associated with mass addition. If a wind stress of magnitude S acts on an ice pack of compactness N , the question arises whether the stress to be used in equation (5) should be S or $S \cdot N$ or $S \cdot f(N)$, where f is some empirically obtained function. It seems that much of the stress $S(1 - N)$ acting on open water will be transmitted from the water to the ice--at least when N is near unity--so that effectively all of S acts on the ice. But then one must take care in characterizing the water stress. And doesn't this imply that the mass of the upper water in leads should be included in the momentum equation for the ice?

The quantity σ_{ij} is the vertically integrated, two-dimensional stress tensor whose units are force per unit length. This is the quantity which determines what we have been calling mechanical behavior; it is the subject of the next section and of four of the succeeding notes.

If equation (5) is to be integrated numerically, there is no particular difficulty in retaining the acceleration, but this term is usually negligible. A typical value of surface stress is 1 dyne/cm²; of concentration, 200 gm/cm². Du_i/Dt is of the order of

$$\frac{1 \text{ cm/sec}}{1 \text{ day (or } 10^5 \text{ sec)}} = 10^{-5} \frac{\text{cm}}{\text{sec}^2} .$$

Hence,

$$\frac{m \frac{D\bar{u}}{Dt}}{S_i}$$

is of the order of 2×10^{-3} , and the acceleration can be neglected. This matter is said to be discussed by Buynitskiy (1951). The primary balance, then, is between body forces, surface stress, and the gradient of the stress tensor.

Of course, when two floes collide, the time scale is very much shorter than one day, and these small-scale accelerations are almost certainly significant in the processes of floe failure and pressuring.

For a review of work based on the momentum equation, especially of various formulations of air and water stress, refer to Campbell (1965).

We now have the system of equations (1) and (5) to be solved for m and \bar{u} , or equations (2) to (5) to be solved for m , H , N , and \bar{u} . Either system of equations requires proper boundary conditions, and this subject has received little attention. The boundary condition on velocity probably varies, depending primarily on compactness, between no slip and no velocity gradient perpendicular to the boundary (corresponding to zero shear stress).

The only work on a system of equations including both mass and momentum balance is that by Doronin (1970) mentioned above.

THE CONSTITUTIVE EQUATION

In order to integrate the momentum equation, we need a constitutive equation which relates the stress tensor to other variables in the problem. Our ignorance of this constitutive equation has been the greatest single

stumbling block to the understanding of ice dynamics, and it is this equation which is the subject of five of the six remaining notes in this Bulletin.

To the extent that the mechanical behavior of pack ice depends physically on pack-ice properties, the constitutive equation depends functionally on them. Alternatively, if we assume that the properties are determined by past strain, the constitutive equation can be taken to depend on strain history instead of on properties explicitly. These two formulations simply provide different descriptions of the same phenomenon. However, the explicit dependence on properties seems the more appealing approach for three reasons.

First, dependence on properties is physically more basic. The reason that non-Newtonian fluid dynamicists must turn to strain history is the difficulty in quantifying such properties as tangled-ness and orientation of long-chain molecules. Characterizing pack properties may be more straightforward. Second, knowledge of some properties would be useful in other aspects of the problem. Third, the constitutive equation probably depends on thermal history and perhaps even on tidal history as strongly as on strain history. Keeping track of all that history is likely to be more cumbersome than knowing several properties.

The assumption that the stress tensor σ_{ij} depends on the strain-rate tensor

$$e_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

leads to several important conclusions:

i) The stress tensor must be an isotropic tensor-valued function of the strain rate tensor. This result, whose proof is given, for instance, by Leigh (1965), follows from the principle of material objectivity which is a fundamental axiom of modern mechanics. As a consequence of this result, the constitutive equation

$$\sigma_{ij} = \eta_{ijkl} e_{kl} \quad ,$$

sometimes proposed for a "linear, anisotropic fluid," cannot represent a real material.

ii) The most general relation between the (two-dimensional) stress tensor and the strain-rate tensor is

$$\sigma_{ij} = A(E_1, E_2) \delta_{ij} + B(E_1, E_2) e_{ij} \quad (6)$$

where A and B are arbitrary scalar-valued functions, and E_1 and E_2 are the two invariants of the strain-rate tensor. This important result was brought to the attention of ice dynamicists by Glen in the note reprinted in this Bulletin.

When a nonlinear form of (6) applies, the material is said to exhibit stress-induced (or strain-rate-induced) anisotropy. An existing state of stress causes the material to respond differently to an incremental stress applied in different orientations.

This brings us to the point of departure of the remainder of this Bulletin. Solomon derives a constitutive equation from a mechanistic model. Four other notes discuss phenomenological approaches to the constitutive equation: Campbell and Rasmussen describe their recent numerical experiments, while Evans, Glen, and Rothrock each consider the implications of more general forms of the constitutive equation.

LATEST EXPERIMENTS WITH ICE RHEOLOGY

by

W. Campbell and A. Rasmussen
U.S. Geological Survey
Tacoma, Washington

We have developed a general ice dynamics model in which the internal ice stress can be treated in a variety of ways. The flow equation includes (in addition to an ice-stress term) an air-stress term, a Coriolis term, and a skin-drag water-stress term that depends on the ice velocity (relative to a stationary ocean). Realizing that it is impossible to verify any model until AIDJEX takes place, we have decided to try a series of assumptions on the form of the internal ice-stress term in our model, so that by obtaining and comparing solutions for the same air-stress driving field we can gain insight into the probable rheological properties of the ice canopy before AIDJEX.

As a second internal ice-stress formulation, we have assumed that (1) at any point in the ice pack the viscosity depends only on the divergence value, and (2) at that point the viscosity applies equally in all directions. Thus the region is made up of areas of convergence and areas of divergence in which the material is isotropic but the value of viscosity is, respectively, large or small. We refer to this model as R-2 (second rheology).

We have obtained a series of steady-state solutions for a segment of free ocean. The effect of the boundary conditions on the open-ocean patch is eliminated by choosing a sufficiently large numerical solution

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Spacecraft Oceanography Project (NOD-NASA), Sea Ice Branch (NOO),
and the U.S. Coast Guard.*

region so that boundary conditions specified at its periphery (no slip, as in Reed and Campbell, 1962) in no way affect the solution within the patch; that is, we get the same solution in the patch no matter what boundary conditions we impose at the periphery of the solution region. We are currently attempting to increase the numerical efficiency of this boundary condition elimination.

Model R-2 gives the same value of convergence and divergence as Model R-1* for a given set of viscosity parameters, but the gradients of divergence in the region around an anticyclonic air-stress field are greater in the former. While the gross features of the divergence field are the same for both models for the same air-stress field, many smaller flow features are different. Both models show a smaller convergence value in the Beaufort gyre than in Campbell's earlier model.

**Editor's note: This model is described on p. 4 of AIDJEX
Bulletin No. 1.*

NOTES ON A POSSIBLE CONSTITUTIVE LAW FOR ARCTIC SEA ICE

by

Roger J. Evans
Department of Civil Engineering
University of Washington
Seattle, Washington

INTRODUCTION

A flow law pertinent to the dynamics of arctic sea ice is no doubt a manifestation of macroscopic phenomena and due probably to interaction of separate ice sheets rather than to ice deformation itself. As such, an appropriate constitutive law must presently be obtained by phenomenological means. The success of Campbell's original model suggests that a viscous type model may indeed be appropriate; certain features of the Campbell model, however, are open to improvement. This note is prompted by discussions at meetings of the Numerical Modeling Committee, by John Glen's recent note to the AIDJEX Committee, and by helpful discussions with Drew Rothrock on the subject.

ISOTROPIC VISCOUS MODEL

Suppose we wish to construct the most general isotropic constitutive law of the form

$$\tau_{ij} = \tau_{ij}(\dot{e}_{kl})^* .$$

This work was supported by the Office of Naval Research Contract N0014-67-A-0103-0007, NR 307-252, with the University of Washington, Department of Atmospheric Sciences (N. Untersteiner, Principal Investigator).

*Index notation is used with Latin indices taking on values 1 or 2.
 τ_{ij} is stress, \dot{e}_{ij} strain rate.

In two dimensions, the general form of this law is

$$\tau_{ij} = F_1 \delta_{ij} + F_2 \dot{e}_{ij} \quad (1)$$

where F_1 and F_2 are both functions of independent invariants of \dot{e}_{ij} , say \dot{e}_{mm} and $\dot{e}_{mn}\dot{e}_{mn}$ --that is,

$$F_1 = F_1(\dot{e}_{mm}, \dot{e}_{mn}\dot{e}_{mn}) , \quad (2)$$

$$F_2 = F_2(\dot{e}_{mm}, \dot{e}_{mn}\dot{e}_{mn}) .$$

The simplest example of this law is the linear law where

$$\begin{aligned} F_1 &= \lambda \dot{e}_{kk} \\ F_2 &= 2\gamma \end{aligned} \quad (3)$$

(λ, γ are constants).

This is equivalent to Glen's linear law when

$$\gamma = \eta , \quad \lambda + \gamma = \zeta .$$

Campbell's original law used (1) with

$$\lambda + \gamma = 0 .$$

This is evident by noting that, on using the strain rate-velocity equations,

$$\dot{e}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

together with (1) and (3), the divergence of stress is given by

$$\tau_{ji,j} = (\lambda + \gamma)u_{k,ki} + \gamma u_{i,jj} ,$$

that is,

$$\nabla \tau = (\lambda + \gamma) \nabla(\nabla \cdot u) + \gamma \nabla^2 u .$$

Thus, because on contraction, (1) with (3) becomes

$$\tau_{kk} = 2(\lambda + \gamma)\dot{e}_{kk} \quad ,$$

omission of the term $\nabla(\nabla \cdot u)$ in the Navier-Stokes equation is equivalent to considering a fluid with zero viscosity under hydrostatic conditions.

It might be noted here that for incompressible fluids, (1) is replaced by

$$\tau_{ij} = -p\delta_{ij} + 2\gamma\dot{e}_{ij} \quad (4)$$

where $-p$ is a scalar function and $F_2 = F_2(\dot{e}_{ij}\dot{e}_{ij})$. Here $\dot{e}_{kk} = 0$ is the continuity equation, or incompressibility condition, and $-p$ is introduced as an unknown in lieu of the now constant density. p , of course, has the significance of hydrostatic stress associated with zero strain rate.

The constitutive law for a compressible (Stokesian) fluid includes a hydrostatic term while relaxing incompressibility. Its constitutive law is thus

$$\tau_{ij} = (-p + F_1)\delta_{ij} + 2\gamma\dot{e}_{ij} \quad (5)$$

where F_1 and F_2 are given by (2). Because the density is not, in general, constant, the addition of $-p$ requires an additional field equation. Since no such state equation appears to exist for sea-ice, the laws (1) for the compressible and (4) for the incompressible case (if required) appear appropriate for viscous representation.

A PROPOSED ISOTROPIC CONSTITUTIVE LAW

Glen has suggested that an appropriate form of (1) might consider F_1 and F_2 to be functions of \dot{e}_{kk} only, i.e., make viscosity depend on volumetric strain rate. In particular, F_1/\dot{e}_{kk} and F_2 might be

constant functions which change in the neighborhood of a particular value of \dot{e}_{mm} . This is the appropriate invariant form of the Campbell-Rasmussen model and was the form discussed at the Numerical Modeling Committee meeting.

Although there appears to be no a priori reason why a linear viscous model should provide an adequate constitutive representation (other than because of the small velocity gradients occurring), there also appears to be no logical reason to suppose that the response should be sensitive to volumetric strain rate. The response would surely, however, be expected to depend on the presence of open water or thin ice in leads between individual ice floes and, since the viscous model has been shown to provide quantitatively realistic results, a viscous model in which viscosity depends on the areal density or ice concentration appears to be the logical extension of the linearly viscous model. Such a model seems essential since the presence of open water between floes is surely represented in the most simple manner by the areal density which is in turn related to the invariants of strain rather than strain rate. Such ideas were, in effect, expressed by A. Assur at the Hanover meeting as well as being the essence of recent Russian work.

The appropriate constitutive law may be developed by considering isotropic constitutive laws for materials for which stress is postulated to be a function of strain and strain rate.* Thus if

$$\tau_{ij} = \tau_{ij}(\dot{e}_{ij}, e_{ij}) \quad (6)$$

where

$$e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i} + v_{k,i}v_{k,j}) \quad [v_k \text{ is displacement}]$$

is the spatial (Euler) strain measure, the most general isotropic, two-dimensional form of (6) is

$$\tau_{ij} = F_1 \delta_{ij} + F_2 \dot{e}_{ij} + F_3 e_{ij} + F_4 \dot{e}_{ik} e_{jk} \quad (7)$$

*Such materials are called "simple viscoelastic materials"; see, for example, A. C. Eringen's Non-Linear Theory of Continuous Media, McGraw Hill, 1962, Chapter 10.

where the F_k 's ($k=1,2,3,4$) are functions of the 5 joint invariants of \dot{e}_{ij} and e_{ij} , that is,

$$F_k = F_k(\dot{e}_{nn}, \dot{e}_{mn}\dot{e}_{mn}, e_{nn}, e_{mn}e_{mn}, \dot{e}_{mn}e_{mn}) \quad .$$

If we now wish to consider the simplest form of (7) which will describe viscous behavior where viscosity depends on strain invariants only, we may take

$$F_3 = F_4 = 0 \quad , \quad F_1 = F_1' \dot{e}_{kk} \quad , \quad F_2 = F_2'$$

$$F_\alpha' = F_\alpha'(e_{nn}, e_{mn}e_{mn}) \quad , \quad \alpha = 1, 2 \quad .$$

If, in addition, F_1' and F_2' depend only on areal density, say ρ , we may write

$$F_\alpha' = F_\alpha'(\rho/\rho_0) \quad , \quad \alpha = 1, 2$$

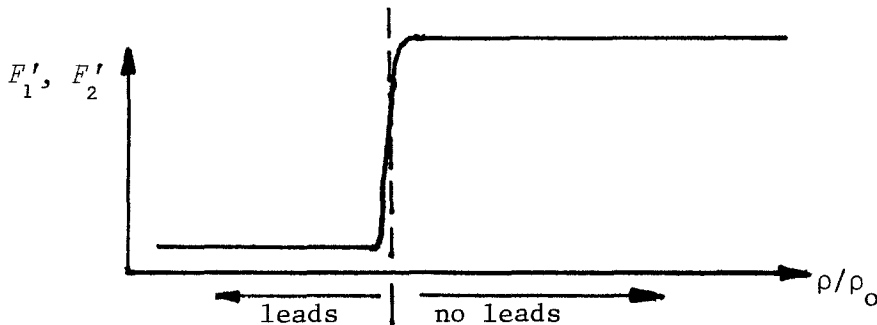
(ρ_0 is initial [undeformed] density and ρ/ρ_0 is related to e_{nn} and $e_{mn}e_{mn}$ by

$$\rho/\rho_0 = [1 - 2e_{mn} + 2\{e_{mn}\}^2 - 2e_{ij}e_{ij}]^{-\frac{1}{2}} \quad)$$

and (6) then takes on the relatively simple form

$$\tau_{ij} = F_1'(\rho/\rho_0) \dot{e}_{kk} \delta_{ij} + F_2'(\rho/\rho_0) \dot{e}_{ij} \quad . \quad (8)$$

Typically, it may be reasonable to try initially functions for F_1' and F_2' which are constants changing in value in the neighborhood of areal densities corresponding to lead-free ice.



THOUGHTS ON A VISCOUS MODEL FOR SEA ICE

by

John W. Glen
Department of Physics
University of Birmingham
Birmingham, England

(Visiting Professor of Geophysics and Atmospheric Sciences,
University of Washington, April-August 1970)

INTRODUCTION

In the planning of experiments and theories about the movement of sea ice, and in particular in the context of a multidisciplinary approach such as AIDJEX, it is useful to form some kind of model for the behavior of sea ice with which observations can be compared. Such a model can then have parameters supplied by experiment, be modified to take account of observed discrepancies, and then, hopefully, be used to predict the behavior of sea ice from synoptic data. One such model which has been considered is the viscous model (Campbell, 1965) in which the ice is supposed to behave like a two-dimensional viscous fluid. Such a model with a single parameter is clearly a rather wild assumption, but the results are sufficiently promising for it to be worth considering how such a model might be made more realistic. Such a suggested modification of the viscous model* has been proposed by Campbell and Rasmussen (private communication). They have proposed that the force equation for sea ice should contain the viscous force via a term of the form

$$K_i \frac{\partial^2 v_i}{\partial x^2} + K_i \frac{\partial^2 v_i}{\partial y^2} = K_i \nabla^2 v_i .$$

This work was supported by the National Science Foundation under a Science Development Grant to the University of Washington.

**Editor's note: Campbell and Rasmussen (p. 12 of this Bulletin) refer to this modified model as R-1.*

This corresponds closely to the term with $K\nabla^2 v_i$ in the corresponding expression for the simple viscous model, but they propose that the value of K_i shall be itself determined by the magnitude of $\frac{\partial v_i}{\partial x_j}$. Basically they propose that this shall have a large value if $\frac{\partial v_i}{\partial x_i}$ is negative, and a small value if it is positive. To prevent the awkwardness of a step function, they propose a form for a smoothed step function to determine this quantity.

To be precise about the magnitude to be inserted, they suggest that it be determined in coordinates which, at the point in question, are principal axes of strain, so that in these coordinates (\bar{x}, \bar{y}) , $\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}$ is equal to zero, and the values of K in the two principal directions are taken as determined by $\frac{\partial \bar{u}}{\partial \bar{x}}$ or $-\frac{\partial \bar{v}}{\partial \bar{y}}$. The value of K_i in any other direction is then determined by an elliptical distribution, so that

$$K_x = K_x^- K_y^- (K_x^{-2} \sin^2 \theta + K_y^{-2} \cos^2 \theta)^{-1/2}, \text{ and}$$

$$K_y = K_x^- K_y^- (K_x^{-2} \cos^2 \theta + K_y^{-2} \sin^2 \theta)^{-1/2}.$$

This proposal introduces into the model the idea that ice behaves differently under motions in which it is compressed and those in which it is extended, but it does so in a way which is not consistent with any clear physical picture of the relation between stress and strain rate; it also ignores certain terms which should be present if such a picture is to be made plausible. Finally, it does not in itself obey normal physical laws for vector relations, and is therefore not invariant with respect to choice of axes. It is the purpose of the present note to elaborate on these objections to the model proposed, and to suggest ways in which a physically more acceptable model might be established.

THE PHYSICAL NATURE OF THE CONSTANT K

In the simple viscous model, the constant K is simply the Newtonian viscosity of the simple viscous fluid. This arises for the following reason:

The relation between stress and strain-rate in a simple Newtonian viscous fluid is that the stress deviator σ' is related to the strain-rate by the equations

$$\sigma'_{ij} = 2\eta \dot{\epsilon}_{ij} \quad .$$

That is to say, the different components of the stress deviator are proportional to the corresponding components of the strain rate. It is understood that the dilatation rate is zero--that is to say, the density of the viscous fluid is constant; thus the trace of the strain-rate matrix is zero, and of course by definition the trace of the stress deviator matrix is also zero.

The force acting on a particle of fluid due to internal stresses is given by the divergence of the stress tensor

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j} \quad ,$$

the Einstein summation convention being used. If we substitute for the stress, we must first put the stress in terms of the stress deviator plus the hydrostatic pressure, and then the first of these terms can be substituted using the viscosity relation to give $2\eta \frac{\partial \dot{\epsilon}_{ij}}{\partial x_j}$.

Finally, we can use the definition of strain-rate, i.e.,

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

to obtain

$$F_i = \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \quad .$$

The second of these terms can be considered as $\frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right)$, and since, as we have said, the flow is considered to be at constant density, it is non-divergent, so this term vanishes. The pressure gradient term must however remain in the momentum equation unless some additional reason exists for its removal. Thus, apart from this pressure term, the internal stress term is of the form assumed with the K equal to the viscosity.

When we come to consider a more complicated case, this simplicity vanishes. The model proposed differs from the simple one in two important respects: (1) K is not a constant but can vary with direction; and (2) it is no longer assumed that the density is constant (this is apparent if in two dimensions at right angles the relation between stress and strain-rate is altered in different proportions from the values they had while preserving the density). The result of the first of these changes is that K ceases to be a scalar quantity and we must consider what it is; the result of the second is that the assumptions we made to get rid of certain terms in the force equation are no longer valid, so that other terms will re-enter and should be taken into account.

Consider the first change first. We now have a situation in which an equation which interrelates vectors has a non-constant multiplying factor; i.e., the relation between the magnitude of the two vectors varies according to the direction of those vectors. This is not an unusual situation; it occurs commonly in crystal physics, where for example the electric field and the electric polarization have different ratios in different crystallographic directions. As is shown in books on crystal physics (e.g., Nye, 1957), if the relation between the two vectors is a linear one, then the interrelating physical entity is a second-rank tensor. Thus if we are to preserve the equation discussed in the introduction, K must be a tensor and we should write the relation

$$F_i = K_{ij} \nabla^2 v_j \quad .$$

We might hope that K_{ij} might be a symmetrical tensor (though we have not proved it); if it is, then it will itself have principal axes, which might be in the directions postulated by Campbell and Rasmussen. In these directions if it had principal values $K_{\bar{x}}$ and $K_{\bar{y}}$, then in any other direction it would have values given by the normal tensor transformation law

$$K_{xx} = K_{\bar{x}} \cos^2 \theta + K_{\bar{y}} \sin^2 \theta,$$

$$K_{yy} = K_{\bar{x}} \sin^2 \theta + K_{\bar{y}} \cos^2 \theta,$$

$$K_{xy} = \left(K_{\bar{x}} - K_{\bar{y}} \right) \cos \theta \sin \theta.$$

This is not the law proposed by Campbell and Rasmussen, which indicates that their K does not transform like a tensor. In particular they have not included the K_{xy} type of term; i.e., they have not explicitly allowed for the fact that unless the direction chosen is a principal axis, the force will not be parallel to the vector $\nabla^2 v$. It would seem a relatively simple matter, however, to use the correct formula rather than the one postulated by Campbell and Rasmussen.

However, the other objection raises more serious problems. The simple equation with K having this simple second-rank tensor form does not follow naturally from simple assumptions about the relation between stress and strain-rate, and the terms in divergence of velocity are not easily disposed of. The relation between stress and strain-rate is a relation, not between two vectors, but between two second-rank tensors. As shown in, e.g., Nye (1957), the kind of entity which enters here is a fourth-rank tensor; i.e., the relationship should be written

$$\sigma_{ij} = 2\eta_{ijkl} \dot{\epsilon}_{kl}$$

in general. In the conventional Newtonian case we do not have to go even to the complications that occur in isotropic, Hookean elasticity (where there are two independent elastic constants, normally taken as any two of the shear modulus, bulk modulus, Young's modulus, and Poisson's ratio). This is simply because the Newtonian liquid is

assumed to be elastic with regard to changes of volume, and the shear alone is reckoned to be viscous; then the only possibility for both isotropy and linearity is for all the coefficients of the tensor to be equal to each other.

However, once we treat sea ice as not being elastic for compressions or extensions of a "hydrostatic" nature--i.e., in our cases increases or decreases of area--we are then faced with having a more complex viscosity, and even for an isotropic material with linear response there will be two viscous coefficients. The best pair to consider is a shear viscosity (because this corresponds exactly to the old viscosity in constant-volume materials) and a bulk viscosity, which relates the rate of increase or decrease of area with the stress required to produce it. Such a viscous model could easily be set up and would have two parameters; but they are not the two parameters of Campbell and Rasmussen's model. How this could be done is considered below. It would also be thinkable to further modify this model to give the bulk viscosity different values according to whether the ice area is increasing or decreasing, or according to whether it was above or below a certain value. Again this would not reproduce the Campbell-Rasmussen model, and it would be essentially a three-parameter model.

Perhaps the best hope for getting something similar to the Campbell-Rasmussen model out of a viscous relation between stress and strain is to assume (1) that the principal axes of stress and strain are parallel, and (2) that in these principal axes σ_x is related to $\dot{\epsilon}_x$ by a K_{xx} that is determined by Campbell and Rasmussen's formula. This would be to assume no "Poisson's ratio" type of terms (which would correspond to having a K_{xy} in our present notation) in these principal axes. Since different values of K are assigned if the principal stresses are positive or negative, it would also carry the consequence that a simple shear stress (where the principal stresses are equal and opposite) would not produce a simple shear strain-rate, but would result in the material increasing fairly rapidly in area.

Even this suggestion, although it would result in giving the terms Campbell and Rasmussen have, would also introduce further terms,

for the direction of the principal stresses will in general vary with position. Although in principal axes at any one point there will not be any $\sigma_{\bar{x}\bar{y}}$, it will not in general be true that there is not a $\frac{\partial \sigma_{\bar{x}\bar{y}}}{\partial \bar{x}}$, and this will introduce a further term when $\frac{\partial \sigma_{ij}}{\partial x_j}$ is calculated. It will also not be true that area is preserved; thus the term in the divergence of velocity must also be taken into account.

Since the model seems to have little physical basis, it is suggested that, rather than make these complications, a physical model based on more comprehensible relations between stress and strain-rate be tried. If, however, actual field experiments can establish what is the relation between stress and strain-rate, then it will be possible to see which law is nearest to reality, and perhaps this two-parameter one should be remembered as a possible model.

A PROPOSED PHYSICAL VISCOUS MODEL

Let us assume that sea ice is viscous, and that it has a viscous relation not only between shear stresses and shear strain-rates, but also between "hydrostatic" (2-dimensional) compressions and area changes. The best way to represent this is to split the strain-rate tensor into two parts by defining a strain-rate deviator

$$\dot{\epsilon}'_{ij} = \dot{\epsilon}_{ij} - \frac{1}{2} \dot{\epsilon}_{kk} \delta_{ij} .$$

Then the relation between this strain-rate deviator and the stress deviator can be left exactly as in normal viscous theory as

$$\sigma'_{ij} = 2\eta \dot{\epsilon}'_{ij}$$

and the relationship between the compressional stress and strain-rate can similarly be put as

$$\sigma_{ii} = 2\zeta \dot{\epsilon}_{ii} .$$

This new constant ζ is the bulk viscosity, a quantity already introduced in conventional physics of fluids to account for the damping of

compressional vibrations. However, in our case we postulate that this is the sole response to a change of area. It is of course open to us to let this constant have different values according to whether the area is increasing or decreasing; although if we tamper with it in this way we must remember what we are doing if we bring the constant through a differential coefficient---this is allowable only if the "constant" is not changing with the variable concerned. We could also arrange that this coefficient varied when the areal density reached some particular value, thus allowing for the fact that if a quantity of sea ice expands and then at once begins to contract again, large forces are not to be expected until its areal density has returned to some particular value.

We can now calculate the force acting due to internal stress. Using the result mentioned in the introduction,

$$\begin{aligned}
 F_i &= \frac{\partial \sigma_{ij}}{\partial x_j} \\
 &= \frac{\partial \sigma'_{ij}}{\partial x_j} + \frac{1}{2} \frac{\partial \sigma_{kk}}{\partial x_i} \\
 &= 2\eta \frac{\partial \dot{\epsilon}'_{ij}}{\partial x_j} + \zeta \frac{\partial \dot{\epsilon}_{kk}}{\partial x_i}
 \end{aligned}$$

We can now substitute for the strain-rate, but since it is the strain-rate deviator that is included, we must write this in terms of the strain-rate and the trace using the first equation on page 24:

$$F_i = 2\eta \frac{\partial}{\partial x_j} \left[\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \right] + \zeta \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_k} \right) .$$

Now two of these terms cancel, because $\frac{\partial^2 u_j}{\partial x_j \partial x_i}$ and $\frac{\partial^2 u_k}{\partial x_j \partial x_k} \delta_{ij}$ are both equal to $\frac{\partial}{\partial x_i} \nabla \cdot \mathbf{u}$. Thus the final equation can be written

$$F_i = \eta \nabla^2 u_i + \zeta \frac{\partial}{\partial x_i} \nabla \cdot \bar{\mathbf{u}} .$$

This is the viscous model proposed in this note, either with two constants η and ζ , or, if the sophistication is desired, with a ζ which varies according to the sign of $\nabla \cdot \bar{u}$.

MORE GENERAL MODELS

If it is necessary to abandon the linear viscous model, even with the modification of two different viscosities for positive and negative areal strain-rates, then it may still be possible to retain initial isotropy as an assumption. If this is so, we can use the general theory of the relation between tensor quantities (see, e.g., Glen, 1958). It can be shown that the most general relation between two second-rank tensors such as stress and strain-rate in two dimensions is of the form

$$\sigma_{ij} = A\delta_{ij} + B\dot{\epsilon}_{ij}$$

where A and B are general functions of the two invariants of the strain-rate tensor. These invariants can be taken as the principal values, or alternatively as $\dot{\epsilon}_{ii}$ and $\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}$. The two special cases of this law considered above are of course of this type. The one which gives the nearest approach to the Campbell-Rasmussen model is expressed in terms of the invariants in principal value form; the one considered in the preceding section has A as a function of $\dot{\epsilon}_{ii}$ only and B as a constant. However, a general law will in general have both A and B as functions of both invariants.

A particular form of the law is that for ideal plasticity. This law has proved useful in glacier flow theory, where also a viscous model was first tried and gave instructive results, showing for example why the center moved faster than the sides, and giving the general reason for marginal crevasses. The ideal plastic law is another approximation to the true law; it has given a further set of insights, including in particular an explanation of shear planes and of extending and compressing flow. It might be expected that an ideal plastic model

of sea ice might explain discontinuous shear planes at points where sea ice encounters obstacles. A difficulty, however, is that existing plasticity theories all assume constant volume (equivalent in 2-dimensions to constant area). If the possibility of areal change is to be retained, a new approach to plasticity theory will be required.

ACKNOWLEDGMENTS

I would like to thank G. A. Maykut for introducing me to this problem, and L. A. Rasmussen for his exposition of the proposed Campbell-Rasmussen model. D. Rothrock suggested the possibility of the ideal plastic approach and contributed useful discussion, and I would like to thank him and also N. Untersteiner for these discussions.

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THE PRESSURE TERM IN THE CONSTITUTIVE LAW OF AN ICE PACK

by

D. A. Rothrock
Research Associate in Atmospheric Sciences
University of Washington
Seattle, Washington

If one sets out to write field equations capable of describing the thermodynamics and the dynamics of the ice pack, the constitutive law governing the pack presents one of the first uncertainties. If the pack is assumed to behave as an isotropic fluid (in two dimensions), the stress tensor is of the form

$$\sigma_{ij} = -p\delta_{ij} + \hat{\sigma}_{ij}(e_{\ell k}), \quad (1)$$

where p is the pressure, δ_{ij} is the Kronecker delta and $\hat{\sigma}_{ij}$ is an isotropic tensor function and must vanish when the strain-rate tensor,

$e_{\ell k} = \frac{1}{2} \left(\frac{\partial u_{\ell}}{\partial x_k} + \frac{\partial u_k}{\partial x_{\ell}} \right)$, vanishes. For a linear viscous fluid,

$$\hat{\sigma}_{ij} = 2\eta e'_{ij} + \zeta e_{\ell\ell} \delta_{ij} \quad (2)$$

where $\eta > 0$ and $\zeta > 0$ are respectively the viscosity and the bulk viscosity of the fluid, and

$$e'_{ij} = e_{ij} - \frac{1}{\delta_{\ell\ell}} e_{\ell\ell} \delta_{ij}$$

is the deviatoric strain-rate tensor. Whether a linear viscous fluid is termed Newtonian for all η and ζ or only when $\zeta = 0$ or $e_{\ell\ell} = 0$

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is a moot point, since Newton only proposed in words that fluid resistance be considered proportional to velocity differences ([3], p. 6).

It is the purpose of this note to discuss the consequences of the retention or omission of the term $(-p\delta_{ij})$ in equation (1).

AN EXAMPLE

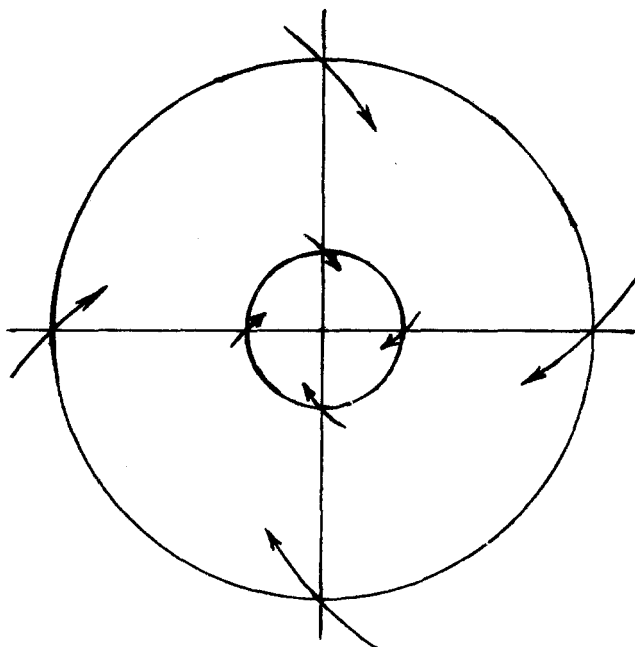
Consider the equiangular spiral prescribed in cylindrical coordinates (r,θ) by the velocity components (u_r, u_θ)

$$u_r = -U r$$

$$u_\theta = -k U r$$

so that $|\bar{u}| = U(1 + k^2)^{\frac{1}{2}} r$.

Let U be a positive constant.



The divergence of this velocity field is everywhere equal to $-2U$. Furthermore, the strain rate tensor is isotropic and independent of r and θ , so that the gradient of $\hat{\sigma}_{ij}$ in the momentum equation is zero, and the pack offers no resistance to this converging motion by means of $\hat{\sigma}_{ij}$. This is so whether the dependence of $\hat{\sigma}_{ij}$ on $e_{\ell k}$ is linear or nonlinear.

In Campbell's (1965) model of the ice pack as a viscous material, the bulk viscosity ζ in equation (2) is taken to be zero, and the pressure term in equation (1) is not included. Campbell notes that the model predicts too rapid a convergence in the Beaufort Sea. Although the gyre in Campbell's model is not equiangular, it exhibits an ice speed which increases along a straight line through the center of the gyre, roughly in proportion to the distance from the center, as in the analytical example described above. It is suggested here that it is a pressure gradient and not viscous stress gradients that should oppose and retard the large convergence observed in that model.

The mathematical reason for the high convergence in Campbell's (1965) model is that the omission of pressure from the stress tensor allows the two scalar momentum equations to be solved for the velocity field without reference to the continuity equation which, for constant mass per unit area, relates the divergence of the velocity field to the rate of production of ice by thermal processes.

It might be mentioned here that Doronin (1970) uses a constitutive law which omits a pressure term, and adopts viscosity proportional to the compactness (areal fraction of ice cover) which is a dependent variable. His calculation also includes thermal effects which change the thickness of existing ice and permits ice production on open water. It is not possible to evaluate the relative merit of any particular aspects of his constitutive law. However, calculations which include thermal effects reproduce the observed ice conditions more closely than those which include no thermal effects.

That the above argument is even more transparent in rectangular geometry has been pointed out by H. Solomon. Consider the velocity $(u,v) = (-Cx, 0)$ in Cartesian coordinates (x,y) . Again, the convergence of the velocity field is C everywhere, and the strain rate tensor is constant so that no stress gradients can develop to oppose the motion. Again, this is so for linear or non-linear isotropic fluids.

ANOTHER EXAMPLE

We now consider a flow in which the relative velocities of points within a finite area are zero. In particular, suppose a steady on-shore wind has pushed the pack against the shore until the ice is quite compact and has come to rest but remains under the influence of the steady wind. The strain-rate tensor is now identically zero, and $\frac{\delta \hat{\sigma}_{ij}}{\delta x_j}$ is also zero

and cannot balance the wind stress. It is the pressure gradient $-\frac{\delta p}{\delta x_i}$ which provides the necessary balance just as it balances the gravitational body force acting on a static volume of fluid.

CONCLUSION

Pressure gradients are undoubtedly present in the ice pack. The scales over which they are important remain to be determined. It should not be argued that the simplicity of the above examples is cause to ignore their implications. It hardly seems sound to expect that a constitutive law which fails to describe simple motions can accurately describe more complicated motions.

The most troublesome consequence of including pressure in the stress tensor is the necessity of relating pressure to other variables. If it is a state variable, the other state variables to which it is related are not obvious at present. Possibly the equation of state involves pressure and properties related to the spatial distribution and energies of the fundamental elements of the medium, namely slabs of sea ice. The energies involved might be kinetic, as is suggested by H. Solomon, or strain energy or both.

ACKNOWLEDGMENTS

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A STUDY OF ICE DYNAMICS
RELEVANT TO AIDJEX

by

Harold Solomon
Department of Environmental Sciences
Tel Aviv University
Ramat Aviv, Israel

(Consultant to AIDJEX during Summer 1970)

I. GENERAL COMMENTS ON AIDJEX

The purpose of AIDJEX is to conduct basic research into the dynamics of the pack ice in the Arctic Ocean. This will be carried out by means of coordinated theoretical and field investigations. The field work is to consist primarily of simultaneous measurements of position and of the forces acting on the ice at several points in the ice pack. The theoretical modeling effort is of central importance, as the theory is the only useful guide to the design of a meaningful field experiment.

Theoretical models of ice drift which have been published up to the present time are mainly based on the equation of motion of a single ice floe. The earliest models, such as were considered by Ekman and by Rossby and Montgomery, attempted to treat only the motion of single ice floes and ignored their mutual interaction. Later models have attempted to treat the motion of an interacting ice pack by adding some kind of frictional term to the equation of motion of a single ice floe. Sverdrup introduced a frictional drag proportional and antiparallel to the velocity. This model is inadequate since, as pointed out by Reed and Campbell, the ice can be accelerated as well as decelerated by interactions between ice floes. Later authors, apparently starting with

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Ruzin, have suggested that the ice be considered as a thin layer of viscous liquid having a high horizontal eddy viscosity. To accomplish this, a horizontal eddy viscosity term having the form of a Laplacian is added to the equation of motion of a single ice floe. A numerical calculation of the steady state wind-driven ice drift was performed for the entire Arctic Ocean by Campbell (1965) using this equation. A further development of this model by Doronin (1970) uses a viscous coefficient proportional to the compactness of the ice, and adds a fluid-type equation of motion is still basically the equation of a single ice floe. Another extension being pursued by Campbell and Rasmussen (work in progress) has a coefficient of eddy viscosity that depends on whether the ice is converging or diverging.

The only published model to date which derives an equation directly including the effect of the interactions of the ice floes is the one-dimensional model by Timokhov (1967). This author is still in the process of reading his paper, but the model apparently does not duplicate what is presented in Part II of this report.

The "viscous" model seems to be the motivation behind the present design of AIDJEX; therefore, it is appropriate to discuss some features of this model in detail. The basic equation is obtained by adding a stress term to the equation of motion of a single ice floe:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = \vec{F} + \nabla \cdot \overleftrightarrow{\tau} \quad (1)$$

Where \vec{v} is the horizontal vector velocity of the ice, ∇ is the horizontal gradient operator, $\overleftrightarrow{\tau}$ is the so-called internal ice stress, and \vec{F} is the sum of surface, body, and inertial forces acting on the ice. In the model of Campbell (1965), it is assumed that the stress is proportional to the velocity shear in the "fluid," so that

$$\nabla \cdot \overleftrightarrow{\tau} = K \nabla^2 \vec{v}, \quad (2)$$

where K is an effective horizontal eddy viscosity. In the AIDJEX plan, a slightly more general concept is employed; it is proposed to calculate

$\nabla \cdot \vec{\tau}$ directly by measuring all of the other terms in (1) and to relate this empirically to the divergence of the rate of strain of the ice pack, to obtain the "divergence of a constitutive law" for the ice pack.

This existing concept of the nature of the ice pack and the appropriate means of describing its dynamics is unclear. Equation (1) was not derived rigorously, nor does it correspond exactly to the correct governing equation of any known physical system which might be taken as the basis for a model of the ice pack. There are at least three possibilities for such models, which seem to be confused in the minds of the ice dynamicists, and none of which are correctly described by equation (1):

1. A two-layer ocean, in which the bottom layer is a standard homogeneous ocean model (representing the liquid ocean) and the top layer is vertically rigid and has a large horizontal eddy viscosity. In this case the upper layer must also satisfy a continuity equation and contain a variable water pressure (in the open leads) or free surface height. It must also satisfy a continuity equation, and be solved for simultaneously with the lower layer, not only because of the frictional coupling but because of the stretching or shrinking of vortex tubes associated with the convergence and divergence in the upper layer. The frictional term should probably include a term of the form $(K+K')\nabla(\nabla \cdot \vec{v})$, where K' is a coefficient of "eddy" bulk viscosity, in addition to the $K\nabla^2 \vec{v}$ term. This should help provide the stiffness associated with the horizontal incompressibility of the ice floes.
2. A "fluid" representing the aggregate of ice floes, which are analogous to "molecules." In this case there must be a continuity equation such as is used by Doronin (1970), and a dependent variable analogous to pressure to represent the direct force of ice floes resisting compression. It is easily shown that this "pressure" must have both scalar and vector parts, since the force arising from frontal collisions of ice floes is not irrotational in general (see Appendix 1). If the floes collide elastically, there must also be a dependent variable analogous to temperature, representing the average

perturbation kinetic energy of motion of individual floes. Along with this variable there must be an equation expressing the conservation of "heat" and relating its rate of production to other dependent variables.

3. A layer of solid, or viscoplastic, material rather than fluid. This is what some ice dynamicists seem to have in mind when they speak of a constitutive law. There will be some difficulty in joining this layer to the ocean because of continuity considerations. Note that the upper two meters of the ocean is actually a discontinuous medium; when the ice floes are together they are unable to converge, but when separated they can converge and as much water will be squeezed out of the leads as if the ice were not present and the same velocity gradient were imposed on this layer. The ice and liquid water have vastly different constitutive properties, and it is not at all clear that the mixture should even obey the same kind of law as either of the constituents. Attempts to find stress-strain laws in granular media have not always been successful (Evans, personal communication).

Thus, there does not seem to be any single clear concept of just what the ice pack is being modeled as.

In the case of elastic or plastic materials, there is experimental evidence which justifies the use of stress-strain relationships. In the case of a fluid, in addition to the experimental evidence there is a theoretical concept in which the viscous force is assumed to arise from the interchange of molecules between fast and slow-moving portions of the fluid, with the mean free path being very small compared to the typical scale of variation of the mean fluid velocity. Hence it is obviously reasonable to assume that the viscous stress is proportional to the shear (or rate of strain). Neither of these justifications exists in the case of the ice pack. The model in Part II of this report indicates that for convergent motion the compressional "stress" may be proportional to the square of the strain rate and depend on other factors as well (assuming inelastic collisions).

In short, it is not clear that there is, or should be, any easily obtainable stress-strain relation which has either fundamental physical significance or reliable practical usefulness. There is only the negative justification that we do not know what else to do.

II. A ONE-DIMENSIONAL MODEL OF ICE FLOE INTERACTIONS

In Part I the deficiencies of the "viscous liquid" model presently used to do numerical ice drift calculations were discussed. The most immediately obvious defect in Campbell's application of this model is the rapid convergence of the ice in the Beaufort Sea, arising from the lack of a continuity equation for the ice and the failure of the viscous term to adequately represent the rigidity and finite size of the ice floes. In this section an attempt will be made to derive a set of equations for ice motion which takes these factors into account.

Let us assume that at the beginning of a freezing season (about 10^7 seconds long) a region with a typical length scale of 1000 km has an average lead width on the order of 1 meter and an average floe width on the order of 100 meters. If the ice floes are perfectly rigid, the convergence over this season (when no melting takes place) is limited to

$$\frac{\partial u}{\partial x} \geq -10^{-9} \text{ sec}^{-1} ;$$

the converging component of velocity cannot vary by more than $0.1 \frac{\text{cm}}{\text{sec}}$ in 1000 km. If the average floe size is more like 1 kilometer, then the converging component of velocity cannot vary by more than $0.01 \frac{\text{cm}}{\text{sec}}$ in 1000 km. It is assumed that if new cracks open in winter the new floes will have to diverge in the process, so that cracking cannot create the possibility of additional convergence. Note that this is a limitation on $\frac{\partial u}{\partial x}$, not on $\frac{\partial^2 u}{\partial x^2}$; therefore, this condition cannot

be met by including a viscous term in the momentum equation, no matter how high one chooses the effective coefficient of viscosity. It also cannot be met in a purely dynamical model unless the model is time-dependent. Convergence can only be allowed in a steady state model if there is a local ice sink, which requires heating.

What follows is a crude attempt to derive a set of equations that incorporate the feature of finite, rigid ice floes. The derivation is not mathematically rigorous; it is hoped, however, that the forms of the terms that have been obtained are essentially correct, and that the effect of using crude assumptions will be that certain parameters, such as the average size of the ice floes, will have to be treated as merely empirical parameters, rather than as the actual average size of the ice floes. Perhaps someday someone will be able to do the derivation rigorously. Meanwhile, it is hoped that what has been done here might turn out to be the beginning of a meaningful contribution to the theory of ice drift. For simplicity, the initial work presented here has been limited to a one-dimensional model; velocities as well as derivatives are limited to one dimension.

In the real ocean, the ice floes come in a variety of sizes and shapes. However, such an assortment of real ice floes would be difficult to treat mathematically. To permit the derivation to proceed without introducing extensive mathematical difficulties at this time, a crude assumption will be introduced: that the real assortment of sizes can be represented mathematically by an equivalent system in which the floe widths are all equal to the average value σ . The symbol σ has been chosen to emphasize the empirical nature of this variable and the resulting analogy with the interaction cross section of particle physics. In this case, σ is the average distance over which external forces are effectively averaged by the rigidity of the ice floes. σ can increase in time as floes stick together. We also need to define d , the average width of the leads. d must be allowed to vary in x as well as t , since leads will be smaller on the average in a region that has been compressed.

In order to focus attention on the mutual interaction of the ice floes, only the momentum of the aggregate of ice floes is being considered. Any effect of the water in the leads on the motion of the ice is considered a part of the water stress, which in turn is incorporated into the sum F of all surface and body forces acting on the ice.

Consider the momentum balance on a length Δx of the ice pack, as shown in Figure 1 below.

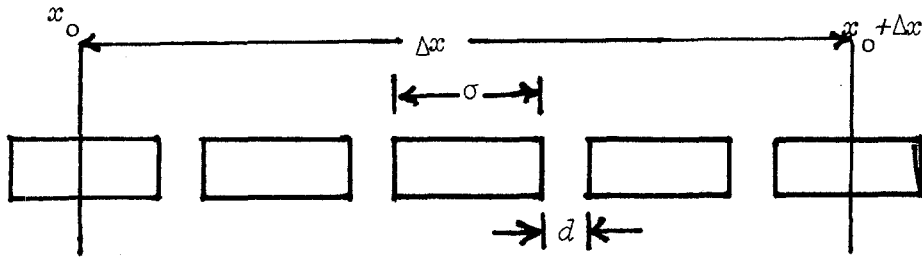


Figure 1

The number of ice floes within the interval is $\frac{\Delta x}{\sigma+d}$. The momentum (per unit of length in the direction perpendicular to the paper) contained within the interval at time t_0 is

$$\left(\frac{\Delta x}{\sigma+d} MU \right) (t_0) , *$$

where M is the mass of a single ice floe.

$$M = \rho_i I \sigma ,$$

where ρ_i is the density of the sea ice and I is its thickness. Since the formation of pressure ridges is being ignored in the present work, I can be taken to be a constant. Any realistic model would have to account for pressure ridge formation, and I (perhaps representing the average thickness including pressure ridges) would have to be a function of x and t . The momentum in the interval at time t_0 can therefore be written as

$$\rho_i I \Delta x \left(\frac{\sigma}{\sigma+d} U \right) (t_0) .$$

*Ed's note: This should be read as " $\left(\frac{\Delta x}{\sigma+d} MU \right)$ evaluated at time t_0 ."

At time $t_0 + \Delta t$, this has changed to

$$\rho_i I \Delta x \left[\frac{\sigma}{\sigma + d} U \right] (t_0 + \Delta t) .$$

An ice floe crossing the plane $x = x_0$ brings momentum into the interval at the rate $+\rho_i I U^2(x_0)$. However, the plane is only straddled by an ice floe for the fraction $\frac{\sigma}{\sigma + d}$ of the total time. Since it has been decided that only the momentum of the ice floes themselves is of interest in the present model, the average rate of momentum transfer across the plane at x_0 is $+\rho_i I \sigma \left[\frac{1}{\sigma + d} U^2 \right] (x_0)$. It will be found convenient to keep the ratio $\frac{\sigma}{\sigma + d}$ intact, so this will be written as $+\rho_i I \left[\frac{\sigma}{\sigma + d} U^2 \right] (x_0)$. The momentum transfer in the time Δt is $+\rho_i I \Delta t \left[\frac{\sigma}{\sigma + d} U^2 \right] (x_0)$. The transfer across the plane $x = x_0 + \Delta x$ in time Δt is $-\rho_i I \left[\frac{\sigma}{\sigma + d} U^2 \right] (x_0 + \Delta x)$, since positive U carries momentum out of the interval.

The impulse delivered to a single ice floe by the sum F of the forces (per unit of length) is $F \sigma \Delta t$. This must be multiplied by the number of floes $\frac{\Delta x}{\sigma + d}$, giving $\frac{\sigma}{\sigma + d} F \Delta x \Delta t$ as the total impulse delivered to the interval in time Δt .

Momentum is transferred into or out of the interval when an ice floe straddling either of the planes $x = x_0$, $x = x_0 + \Delta x$ collides with another floe which is entirely either in or out of the interval. Note that collisions which take place between two floes both of which are entirely within the interval do not change the total momentum in the interval.

The difference in speed between two successive ice floes is $\frac{\partial u}{\partial x} (\sigma + d)$ (on the average). An assumption about the dynamics of the

system is now necessary in order to calculate the average time required for a collision to occur. If momentum transfer takes place mainly through collisions, so that the speeds of the two floes do not change as they move toward collision, the average time required for a collision to occur is

$$\frac{d}{-(\sigma + d) \frac{\partial u}{\partial x}} .$$

The - sign is needed because $\frac{\partial u}{\partial x}$ must be negative in order for a collision to occur. In Appendix 2 it is shown that this is not necessarily true if collisions play a relatively minor role in the momentum balance. Since the primary interest in the present work is in the case in which collisions are important, this assumption will be used here. Then the average number of collisions undergone by an ice floe in time Δt is

$$-\left[\frac{\Delta t}{d} \right] \frac{(\sigma + d) \frac{\partial u}{\partial x}}{(\sigma + d) \frac{\partial u}{\partial x}} \Delta t = - \frac{(\sigma + d) \frac{\partial u}{\partial x}}{d} \Delta t .$$

However, an ice floe only straddles the boundary of the interval for the fraction $\frac{\sigma}{\sigma + d}$ of the total time, so the number of collisions involving a floe on either plane ($x = x_0$ or $x = x_0 + \Delta x$) in time Δt is:

$$-\frac{\sigma}{(\sigma + d)} \frac{(\sigma + d) \frac{\partial u}{\partial x}}{d} \Delta t = - \frac{\sigma \frac{\partial u}{\partial x}}{d} \Delta t .$$

An assumption must be made about the nature of the collisions between ice floes in order to estimate the momentum transfer per collision. It must be emphasized that no meaningful theory of the large-scale flow of sea ice can be derived until the small-scale interactions are much better understood than at present. However, it is possible to see how such a theory might go by making an assumption about the interactions. The simplest such assumption is that the ice floes collide inelastically. Inelastic collisions have been frequently observed in the Arctic Ocean, although elastic collisions have also been reported. An assumption of elastic collisions would require us to introduce an effective ice "temperature" and "pressure" to account for the more or less random

"thermal" motion set up by the rebounding of the ice floes. Assuming that collisions are inelastic, conservation of momentum requires the change in momentum of each floe during the collision to be

$$\frac{1}{2} \left(\rho_i I \sigma \right) \left(\frac{\partial u}{\partial x} \sigma \right)$$

in absolute value, where $\rho_i I \sigma$ is the mass of each floe and $\frac{\partial u}{\partial x} \sigma$ is the difference in speeds of the two floes at the time of collision. Considering the plane $x = x_0$, if $\frac{\partial u}{\partial x} < 0$ (necessary for a collision to occur), a collision always adds positive momentum to the interior of the interval; and on the average, exactly one of the two floes is within the interval. Therefore, at x_0 the change in momentum within the interval per collision is

$$- \frac{1}{2} \rho_i I \sigma^2 \frac{\partial u}{\partial x} \left(x_0 \right) .$$

At $x = x_0 + \Delta x$, negative $\frac{\partial u}{\partial x}$ causes collisions which transfer positive momentum out of the interval, so that momentum transfer per collision is

$$+ \frac{1}{2} \rho_i I \sigma^2 \frac{\partial u}{\partial x} \left(x_0 + \Delta x \right) .$$

The total transfer of momentum to the interval through collisions in time Δt therefore becomes

$$+ \frac{1}{2} \rho_i I \Delta t \sigma^3 \left[\frac{1}{d} \left(\frac{\partial u}{\partial x} \right)^2 \right] \left(x_0 \right) - \frac{1}{2} \rho_i I \Delta t \sigma^3 \left[\frac{1}{d} \left(\frac{\partial u}{\partial x} \right)^2 \right] \left(x_0 + \Delta x \right) .$$

The complete difference equation is

$$\begin{aligned} \rho_i I \Delta x \left(\frac{\sigma}{\sigma + d} U \right) \left(t_0 + \Delta t \right) &= \rho_i I \Delta x \left(\frac{\sigma}{\sigma + d} U \right) \left(t_0 \right) \\ + \rho_i I \Delta t \left(\frac{\sigma}{\sigma + d} U^2 \right) \left(x_0 \right) &- \rho_i I \Delta t \left(\frac{\sigma}{\sigma + d} U^2 \right) \left(x_0 + \Delta x \right) + \frac{\sigma}{\sigma + d} F \Delta x \Delta t \\ + \frac{1}{2} \rho_i I \Delta t \sigma^3 \left[\frac{1}{d} \left(\frac{\partial u}{\partial x} \right)^2 \right] \left(x_0 \right) &- \frac{1}{2} \rho_i I \Delta t \sigma^3 \left[\frac{1}{d} \left(\frac{\partial u}{\partial x} \right)^2 \right] \left(x_0 + \Delta x \right) . \end{aligned}$$

Dividing through by $\rho_i I \Delta x \Delta t$, collecting terms, taking the limit as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$ and expanding the derivative in the collision term gives:

$$\frac{\partial}{\partial t} \left(\frac{\sigma}{\sigma + d} U \right) = - \frac{\partial}{\partial x} \left(\frac{\sigma}{\sigma + d} U^2 \right) + \frac{\sigma}{\sigma + d} \frac{F}{\rho_i I} + \frac{1}{2} \frac{\sigma^3}{d^2} \frac{\partial d}{\partial x} \left(\frac{\partial u}{\partial x} \right)^2 - \frac{\sigma^3}{d} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2}. \quad (3)$$

Note that the last term has the form of a viscosity proportional to $\frac{\partial u}{\partial x}$. If $\sigma = 1 \text{ km}$, $d = 1 \text{ m}$, and $\frac{u}{x} = 10^{-9} \text{ sec}^{-1}$, the effective "viscous coefficient" is about 1, which gives a very weak "viscous" interaction compared to that which was used by Campbell (1965). If $\sigma = 10 \text{ km}$, this becomes 10^{+3} , which is still rather small. Hence, this model gives a relatively weak "viscous" type of interaction when the average lead width is on the order of 1 meter. This makes sense because there is really nothing to resist convergence as long as the floes are separated by leads; it is only when they are pressed together that they are unable to converge further. When $d \rightarrow 0$, the effective viscosity coefficient becomes large. However, it still cannot be said that a model has been produced which absolutely requires $\frac{\partial u}{\partial x} = 0$, because we could conceivably have $\frac{\partial^2 u}{\partial x^2} = 0$ and $\frac{\partial d}{\partial x} = 0$ without $\frac{\partial u}{\partial x} = 0$.

Also note that the factor $\frac{\sigma}{\sigma + d}$, equivalent to Doronin's N , appears in every term except the collision term. This expresses the fact that if we are really treating the aggregate of ice floes, rather than a combined upper layer of ice and water, the total momentum in any region is proportional to the amount of ice in that region. The failure of this factor to appear in previous models is a consequence of the failure to specify exactly what the layer of ice is being modeled as. It was pointed out by Martin (personal communication) that the failure to include this factor amounted to using a "Boussinesq approximation" without justification.

Two more equations are needed to complete the system needed to solve for U , σ and d . One is probably in the form of a mass continuity equation, but the form of the third is still uncertain. Preliminary study by this author indicates that it may be necessary to keep track (numerically) of the actual distribution of lead widths in order to

calculate the resulting values of $\sigma(t)$ and $d(x,t)$. For example, one could start with the Poisson distribution

$$P(\eta) = \lambda \eta e^{-\lambda \eta}$$

and normalize so that

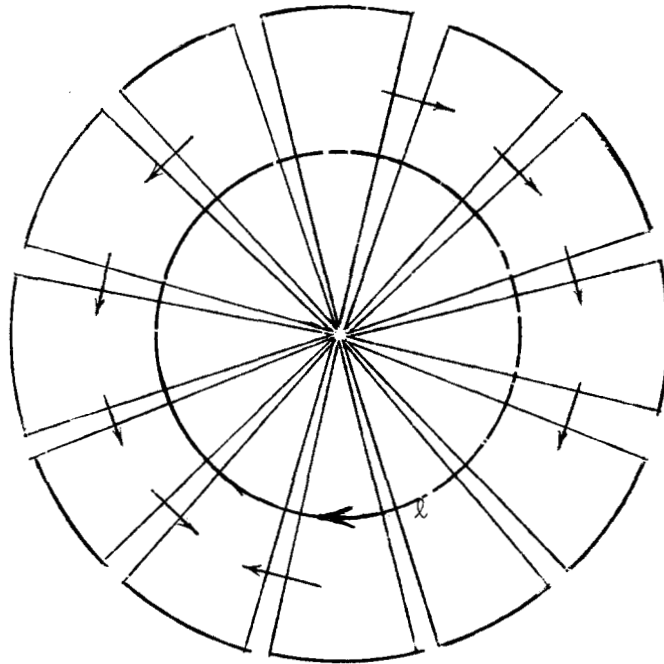
$$d = \int_0^{\infty} \lambda \eta e^{-\lambda \eta} d\eta ,$$

which gives $\lambda = \frac{1}{d}$. However, after time Δt , one could not assume that the new distribution is a Poisson distribution with some new average width d_1 . Rather, if all leads smaller than d_2 close up in Δt , the new distribution would be

$$P(\eta) = \lambda(\eta + d_2) e^{-\lambda(\eta + d_2)} .$$

APPENDIX 1

It is desired to show that the "pressure" force arising from the frontal collisions of ice floes is not irrotational in general. It suffices to show that the force field is not irrotational in one particular case. Consider the system of wedge-shaped ice floes in Figure 2.



Suppose that the floes are set into circular motion (short arrows) with the tangential velocity proportional to $\sin\theta$. The force can be assumed to be proportional to the tangential derivative of the tangential velocity; if some power is involved (in Part II the force was found to be of the form $\frac{\partial}{\partial x} \left[\left(\frac{\partial u}{\partial x} \right)^2 \right]$), the result will be substantially the same. We have to calculate the circuital line integral of this force around the loop ℓ :

$$\oint_{\ell} \vec{F} \cdot d\vec{s} = \int \frac{\partial}{\partial \theta} (\sin \theta) R d\theta + \int 0 \cdot R d\theta$$

converging regions
diverging regions

$$= R \int \cos \theta d\theta$$

converging regions

Since the integral is only taken over regions where $\cos \theta < 0$, it obviously cannot be zero, so

$$\oint_{\ell} \vec{F} \cdot d\vec{s} \neq 0 \quad ,$$

which is what we were trying to prove.

APPENDIX 2

In the case when collisions do not play the major role in momentum transfer between ice floes, the average time between collisions might not be

$$\frac{d}{-(\sigma + d) \frac{\partial u}{\partial x}}$$

Consider the limiting case when U is a function of x only and momentum transfer by collisions is negligible. Ice floe 1 is initially centered at $x = x_1$, and advances to $x = x_2$ at the time of the collision. Ice floe 2 is initially centered at $x = x_1 + (\sigma + d)$, and advances to $x = x_2 + \sigma$ at the time of the collision. Since the times required for the floes to reach the collision are equal, we have

$$\int_{x_1}^{x_2} \frac{1}{U(x)} dx = \int_{x_1 + \sigma + d}^{x_2 + \sigma} \frac{1}{U(x)} dx \quad (4)$$

as our basic equation.

If the floes collide after traveling a distance short compared to σ (Figure 3), (4) becomes approximately

$$\frac{x_2 - x_1}{U(x_1)} = \frac{(x_2 + \sigma) - (x_1 + \sigma + d)}{U(x_1 + \sigma + d)}$$

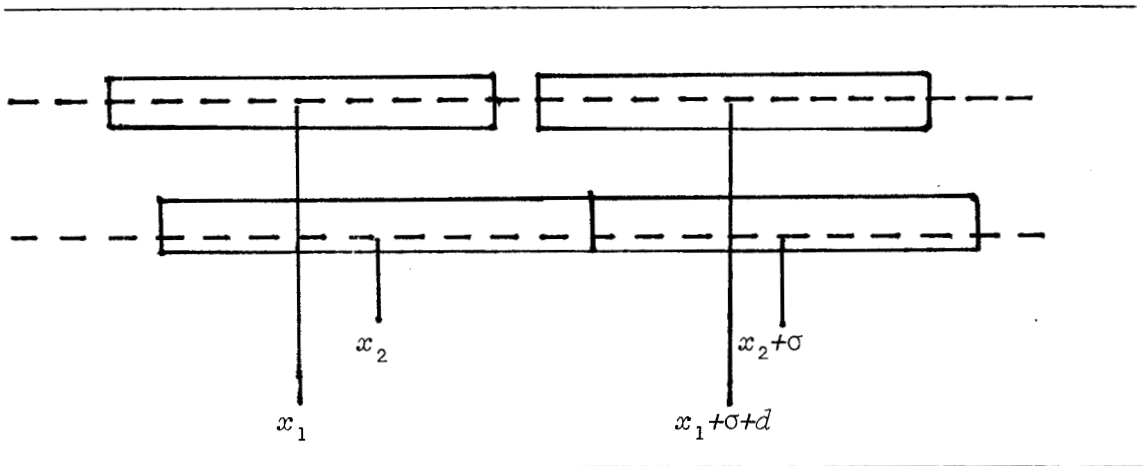


Figure 3

Expanding $U(x_1 + \sigma + d) \approx U(x_1) + (\sigma + d) \frac{\partial U}{\partial x}$, and neglecting higher order terms because U is slowly varying in x , we find that

$$x_2 - x_1 = \frac{U(x_1) d}{-(\sigma + d) \frac{\partial U}{\partial x}},$$

so that the time between collisions is

$$\frac{x_2 - x_1}{U(x_1)} = - \frac{d}{(\sigma + d) \frac{\partial U}{\partial x}},$$

as in the case in which the transfer of momentum takes place mainly through collisions. However, if the flocs travel a distance long compared to σ before colliding (Figure 4), the appropriate form of (4) becomes

$$\int_{x_1}^{x_1 + (\sigma + d)} \frac{1}{U(x)} dx = \int_{x_2}^{x_2 + \sigma} \frac{1}{U(x)} dx$$

$$\frac{x_1 + (\sigma + d) - x_1}{U(x_1)} = \frac{x_2 + \sigma - x_2}{U(x_2)}$$

This time $U(x_2) \approx U(x_1) + (x_2 - x_1) \frac{\partial U}{\partial x}$

$$x_2 - x_1 = \frac{U(x_1) d}{-\sigma \frac{\partial U}{\partial x}}$$

and the time between collisions is about

$$\frac{x_2 - x_1}{U(x_1)} = - \frac{d}{\sigma \frac{\partial U}{\partial x}}$$

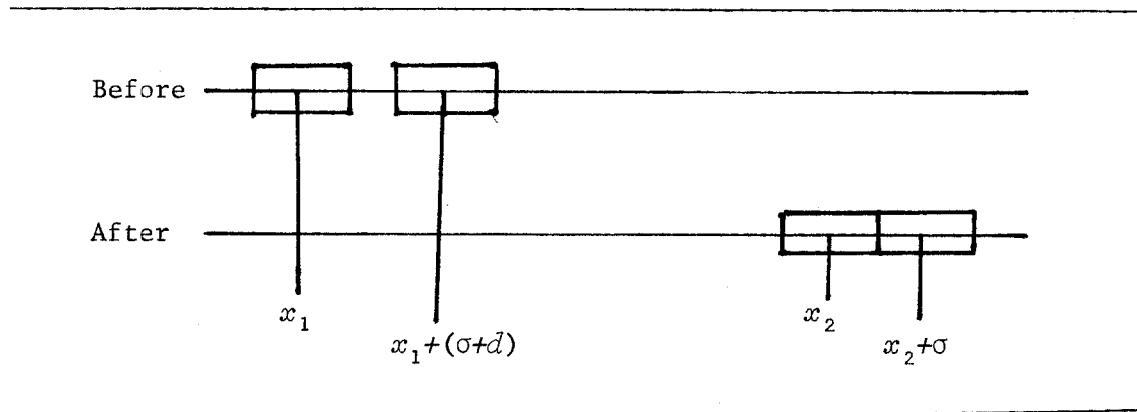


Figure 4

The distinction can be important where d is not small compared to σ . In general, of course, we are somewhere in between the extreme cases treated here, so that the time between collisions is

$$\frac{d}{(\sigma + \epsilon d) \frac{\partial u}{\partial x}}$$

where ϵ is between 0 and 1, and depends on:

1. The relative importance of momentum transfer by collisions, as compared to momentum transfer by other processes.
2. The ratio of the mean free path to the average "interaction cross section" of the ice floes.

The motion is strongly time-dependent. Of course, the various forms derived here are equivalent when $\sigma \gg d$, as in the region of primary interest to AIDJEX.

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TECHNIQUES FOR MEASURING STRAIN RATE

by

Alan Thorndike
Graduate Student in Geophysics
University of Washington
Seattle, Washington

This report summarizes the progress made to date by R. Wills (Graduate Student in Philosophy, University of Washington) and myself towards describing the deformation of portions of the arctic ice pack from series of position data reported by the several drifting stations. Such studies have been undertaken before by Russian scientists,¹ and in this country by Dunbar and Wittmann (1962). Their publications report only the divergence (rate of change of area) of a region. Since the entire strain-rate tensor more completely describes deformation, its study is a proper objective of AIDJEX.

The linear strain-rate tensor is defined by

$$\epsilon_{ij} \equiv \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i=1,2; \quad j=1,2 \quad . \quad (1)^2$$

It has three independent components ϵ_{11} , $\epsilon_{12}=\epsilon_{21}$, and ϵ_{22} , each involving spatial derivatives of velocity. For example, ϵ_{22} represents the rate of change of distance (stretching) between a pair of points lying on the 2-axis. In fact, if the line joining two points makes an angle ϕ with the 1-axis, then the rate of stretching of that line is given by

$$\epsilon_{\phi} = \cos^2\phi \epsilon_{11} + \sin 2\phi \epsilon_{12} + \sin^2\phi \epsilon_{22} \quad . \quad (2)^3$$

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To compute the strain-rate tensor, one proceeds as follows. Rates of stretching are measured for each pair of drifting stations, giving n equations identical in form to equation (2). The only unknowns are ϵ_{11} , ϵ_{12} , and ϵ_{22} , which are determined by a standard least squares technique.

Formally, we wish to solve $\vec{\epsilon}_\phi = \vec{L} \vec{\epsilon}$, for $\vec{\epsilon}$, where

$$\vec{\epsilon}_\phi \Leftrightarrow \begin{pmatrix} \epsilon_{\phi_1} \\ \epsilon_{\phi_2} \\ \vdots \\ \epsilon_{\phi_n} \end{pmatrix}, \quad \vec{L} \Leftrightarrow \begin{pmatrix} \cos^2\phi_1 & \sin 2\phi_1 & \sin^2\phi_1 \\ \cos^2\phi_2 & \sin 2\phi_2 & \sin^2\phi_2 \\ \vdots & \vdots & \vdots \\ \cos^2\phi_n & \sin 2\phi_n & \sin^2\phi_n \end{pmatrix}, \quad \text{and} \quad (3)$$

$$\vec{\epsilon} \Leftrightarrow \begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{22} \end{pmatrix} \quad .4$$

The solution to equation (3) is

$$\vec{\epsilon} = (\vec{L}_t \vec{L})^{-1} \vec{L}_t \vec{\epsilon}_\phi \quad .5 \quad (4)$$

ERRORS

In the least squares approach, each measurement ϵ_{ϕ_i} is thought of as the sum of some ideal value ϵ_{ϕ_i}' and an error discrepancy v_i .

$$\epsilon_{\phi_i} = \epsilon_{\phi_i}' + v_i, \quad i=1, n \quad (5)$$

The restraint that $\sum_{i=1}^n v_i^2$ be a minimum gives enough equations to allow solution of equation (3). The ideal value, $\vec{\epsilon}'_{\phi}$, is simply $\vec{L}^{-1} \vec{\epsilon}$, so that

$$\vec{v} = \vec{\epsilon}_{\phi} - \vec{L} (\vec{L}_t \vec{L}_t)^{-1} \vec{L}_t \vec{\epsilon}_{\phi} . \quad (6)$$

The discrepancies v_i arise from errors in the measurement of the rates of stretching ϵ_{ϕ_i} and from nonlinear variations of $\vec{\epsilon}$ itself within the regions of interest. The latter source of error is inherent in any finite difference approximation of the derivative of a field variable. Such errors are lumped together with and indistinguishable from errors in measurement of ϵ_{ϕ_i} .

In the simple case of a triangular array of stations ($n=3$), there will be no apparent errors v_i . Nevertheless, errors in measurement of the angles of the triangle, θ_i , and the rates of stretching, ϵ_{ϕ_i} , will conspire against confident estimates of $\vec{\epsilon}$. The situation is entirely analogous to fitting a straight line to a set of points in a plane. Since the line will not, in general, pass through all the points, the best line is selected by minimizing the apparent errors. When there are only two points, it is possible to find a line which passes through both points, but confidence in this line is still limited by the errors in defining the two data points.

The manner in which errors propagate through the equations depends on θ_1 , θ_2 , and on $\vec{\epsilon}$ itself. We have no control over $\vec{\epsilon}$, but we are free to select the geometry of the array to our advantage. Figure 1 shows the dependence of the expected error on the geometry of a triangular

array for the special case

$$\epsilon_{11} = .02 \text{ day}^{-1}, \epsilon_{12} = -.02 \text{ day}^{-1}, \epsilon_{22} = -.02 \text{ day}^{-1}, \text{ expected error in } \theta = 1 \text{ degree, expected error in } \epsilon_{\phi} = .001 \text{ day}^{-1} .$$

The surfaces depend on the choice of coordinates in which $\vec{\epsilon}$ is measured. In principal axes, for instance, the error surface reaches its minimum value whenever $\theta_1, \theta_2,$ or $\theta_3 = 90^\circ$. Not knowing the orientation of the principal axes in advance, however, it seems best to select an equilateral array.

For an array of more than three stations, I suspect that the most favorable geometry for measuring $\vec{\epsilon}$ is a regular polygonal array, since it offers an even sampling of ϕ_i and equal length segments. To measure spatial variations, a cross-type array may be more useful. Such an array still samples ϕ_i well while giving greater horizontal coverage.

VARIATIONS IN STRAIN RATE

With a sufficiently dense array, it would be possible to search for variations in the strain-rate tensor with position. One approach would be to compute the strain-rate tensor for subsets of the entire array and then to inspect the results for significant variations. A more efficient use of the data is possible if we are willing to assume that $\vec{\epsilon}$ varies linearly throughout the region of interest.⁶ With this assumption, we may write as equalities:

$$\begin{aligned} \epsilon_{11} &= \epsilon_1 + a_1x + b_1y \\ \epsilon_{12} &= \epsilon_2 + a_2x + b_2y \\ \epsilon_{22} &= \epsilon_3 + a_3x + b_3y . \end{aligned} \tag{7}$$

Substituting these definitions into equation (3), we write a new equation in matrix form: [For typographic reasons, *sine* and *cosine* are abbreviated as *s* and *c* .]

$$(8) \begin{pmatrix} \epsilon_{\phi_1} \\ \epsilon_{\phi_2} \\ \vdots \\ \vdots \\ \epsilon_{\phi_n} \end{pmatrix} = \begin{pmatrix} c^2\phi_1 & x_1c^2\phi_1 & y_1c^2\phi_1 & s2\phi_1 & x_1s2\phi_1 & y_1s2\phi_1 & s^2\phi_1 & x_1s^2\phi_1 & y_1s^2\phi_1 \\ c^2\phi_2 & x_2c^2\phi_2 & y_2c^2\phi_2 & s2\phi_2 & x_2s2\phi_2 & y_2s2\phi_2 & s^2\phi_2 & x_2s^2\phi_2 & y_2s^2\phi_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c^2\phi_n & x_nc^2\phi_n & y_nc^2\phi_n & s2\phi_n & x_ns2\phi_n & y_ns2\phi_n & s^2\phi_n & x_ns^2\phi_n & y_ns^2\phi_n \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ a_1 \\ b_1 \\ \epsilon_2 \\ a_2 \\ b_2 \\ \epsilon_3 \\ a_3 \\ b_3 \end{pmatrix}$$

which has the form $\vec{\epsilon}_\phi = \vec{M} \vec{\tilde{\epsilon}}$. The solution for $\vec{\tilde{\epsilon}}$ is

$$\vec{\tilde{\epsilon}} = (\vec{M}_t \vec{M})^{-1} \vec{M}_t \vec{\epsilon}_\phi , \quad (9)$$

provided $n \geq 9$. The pair (x_i, y_i) is taken to be the midpoint of the line segment used to compute ϵ_{ϕ_i} .

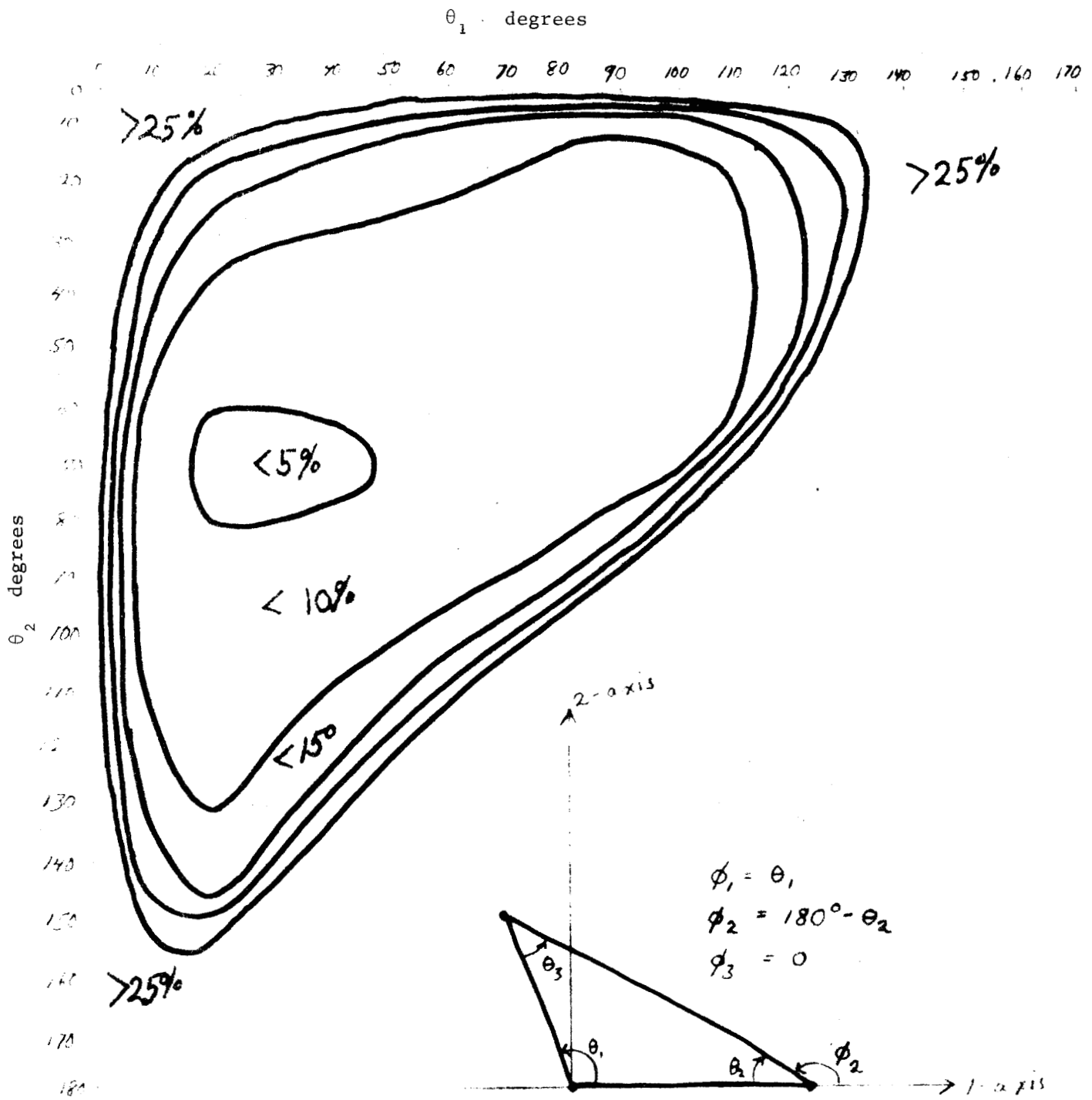


Figure 1. The dependence of σ on the geometry of a triangular array. (see p. 57)

FIGURE 1. The dependence of σ on the geometry of a triangular array. Note that $\theta_3 = 180 - \theta_1 - \theta_2$ implies that θ_3 decreases in the SE direction. σ (a measure of the expected error in computations of the strain-rate tensor) is defined as

$$\sigma \equiv \frac{\sqrt{1/3 \left(\sigma_{\epsilon_{11}}^2 + \sigma_{\epsilon_{12}}^2 + \sigma_{\epsilon_{22}}^2 \right)}}{\text{typical value of } \epsilon_{ij}}$$

where $\sigma_{\epsilon_{ij}}$ is the mean squared error in the computation of ϵ_{ij} . The $\sigma_{\epsilon_{ij}}$ are functions of θ_1 , θ_2 , $\delta\theta$ (expected error in measurement of θ), $\delta\epsilon_\phi$ (expected error in measurements of ϵ_ϕ), and of $\vec{\epsilon}_\phi$. Thus, σ is the average of the expected errors in each of the three components of $\vec{\epsilon}$, divided by the typical value of ϵ_{ij} , here taken as $.02 \text{ day}^{-1}$. In the case shown, $\epsilon_{11} = .02$, $\epsilon_{12} = -.02$, $\epsilon_{22} = -.02 \text{ day}^{-1}$, $\delta\theta = 1^\circ$, $\delta\epsilon_\phi = .001 \text{ day}^{-1}$. The general shape of the error surface is not sensitive to the choice of these quantities, although the precise location of the minimum does shift as $\vec{\epsilon}$ changes. The estimates of $\delta\theta$ and $\delta\epsilon_\phi$ used here probably do not reflect the state of the art in arctic surveying.

NOTES

1. A. V. Bushuyev, N. A. Volkov, Z. M. Gudkovich, and V. S. Loshchilov. "Results of a Soviet Experiment Investigating the Drift and Dynamics of the Arctic Basin Ice Cover" (Rezultaty Ekspeditsionnykh Issledovaniy Dreifa i Dinamiki Ledianogo Pokrova Arkticheskogo Basseina Vesnoi, 1961 g). Leningrad Arkticheskii i Antarkticheskii Nauchno-issledovatel'skii Institut. Trudy. Vol. 257, 26-44. 1967. Translated from the Russian by S. M. Olenicoff and H. Solomon, The Rand Corporation, May 1970.

2. Note that I have assumed that the problem is two-dimensional. Accordingly, the Arctic Ocean has been projected onto a plane tangent to the earth at the North Pole, using the formulae

$$x = \left(\frac{a \cos \theta}{\xi - 1 + \sin \theta} \right) \cos \psi ,$$
$$y = \left(\frac{a \cos \theta}{\xi - 1 + \sin \theta} \right) \sin \psi ,$$

where θ is north latitude, ψ is east longitude, a is the radius of the earth, and $\xi = 2.979$. ξ was chosen to force $-\partial x / \partial \theta$ to be unity at $\theta = 70^\circ \text{N}$. Between 70° and 90°N , $\partial x / \partial \theta$ varies less than 0.5 percent. The amount of distortion introduced by the projection seems tolerable.

3. To show this, use the familiar laws to transform the tensor $\overset{\rightarrow}{\epsilon}$ into a tensor $\overset{\rightarrow}{\epsilon}'$ in a coordinate system whose 1'-axis intersects the 1-axis with angle ϕ . Then $\epsilon_{11}' = \epsilon_\phi$.

4. When strain rate is written as a square matrix (vector), we use a double (single) arrow. Thus,

$$\overset{\rightarrow}{\epsilon} \Leftrightarrow \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix}, \quad \overset{\rightarrow}{\epsilon} \Leftrightarrow \begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{22} \end{pmatrix} .$$

5. General Least Squares Theorem.

Suppose $A = LX$ is a matrix equation where A is known, L is known and does not depend on X , and X is to be found. Let the dimensions be as follows:

$$\begin{aligned} A(n \times m) \\ L(n \times l) \\ X(l \times m) . \end{aligned}$$

When $n < l$, not enough information is available to find X . When $n = l$, the solution is $X = L^{-1}A$ provided L^{-1} exists. When $n > l$, too much information is available, and it is appropriate to seek a solution X which minimizes the residuals $A-LX$.

To be precise, let $\epsilon_{ij} = A_{ij} - L_{ir} X_{rj}$ (summation convention). We stipulate that $E = \epsilon_{ij} \epsilon_{ij}$ must be a minimum. E is a function of the lm quantities $X_{k\ell}$, so our stipulation in fact states the equations

$$\frac{\partial E}{\partial X_{k\ell}} = 0 \Rightarrow \epsilon_{ij} \frac{\partial \epsilon_{ij}}{\partial X_{k\ell}} = 0 .$$

But

$$\frac{\partial \epsilon_{ij}}{\partial X_{k\ell}} = \frac{\partial}{\partial X_{k\ell}} (A_{ij} - L_{ir} X_{rj}) = -L_{ir} \frac{\partial X_{rj}}{\partial X_{k\ell}} = -L_{ir} \delta_{rk} \delta_{j\ell} = -L_{ik} \delta_{j\ell}$$

where we have used the assumption $L \neq L(X)$.

We have these equations: $\epsilon_{ij} = A_{ij} - L_{ir} X_{rj}$ $n \times m$ equations
 $\epsilon_{ij} (L_{ik} \delta_{j\ell}) = 0$ $l \times m$ equations

in the unknowns X_{ij} and ϵ_{ij} , a total of $n \times m + l \times m$ unknowns. The solution is found by eliminating ϵ_{ij} to get

$$\begin{aligned} 0 &= (A_{ij} - L_{ir} X_{rj}) L_{ik} \delta_{j\ell} = (A_{i\ell} - L_{ir} X_{r\ell}) L_{ik} \\ \Rightarrow L_{ki}^t A_{i\ell} &= L_{ki}^t L_{ir} X_{r\ell} \Rightarrow X = (L^t L)^{-1} L^t A . \end{aligned}$$

6. This seems reasonable, since it would require many additional stations to detect 2nd-order variations. However, the extension of this method to higher-order variations is straightforward if the requisite number of stations is available.

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POWER SPECTRUM ANALYSIS OF ICE RIDGES

By: William D. Hibler III ‡
 Leonard A. Le Schack ††

The technique of power spectrum analysis¹ has for a number of years been a useful tool for communications engineers in extracting periodic variations from space or time series. With laser profiles as well as under ice sonar profiles one can apply the power spectrum technique to determine preferred ridge spacings and the amount of ice exhibiting a given periodicity. Dominant ridge spacings may be of considerable interest for dynamical ice modeling. The power spectrum technique can also be used for possibly identifying different ice types by their characteristic frequencies.

A brief formal description of the power spectrum procedure may be given as follows.

Given a spatial series, $X(r)$, the auto-covariance function is defined by

$$C(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \eta(x) \eta(x+\tau) dx \tag{1}$$

where $\eta(x) = X(x) - \bar{X}(x)$.

The power spectrum, $P(f)$, is defined by

$$P(f) = \int_{-\infty}^{\infty} C(\tau) e^{-2\pi i f \tau} d\tau \tag{2}$$

From this definition it can be shown that $P(f)$ is also given by

‡ Research Physicist, U.S. Army Cold Regions Research & Engineering Laboratory, Hanover, New Hampshire

†† President, Development and Resources Transportation Corporation
 Silver Spring, Maryland

$$P(f) = \lim_{T \rightarrow \infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} \eta(x) e^{-2\pi i f x} dx \right|^2 \quad (3)$$

This expression is most useful for interpretation because the right-hand side of Eq. (3) is simply the square of the fourier transform of $\eta(x)$.

From the number of data points used and the number of frequency intervals one can predict the confidence limits of a given spectrum about the "ideal" spectrum assuming a stationary gaussian process. We will not go into the details of the analysis here but simply remark that various techniques may be used for satisfactorily estimating the true power spectrum and thus the true fourier transform of a given sample, which therefore makes this procedure useful.

One example of the use of this technique is the analysis of laser profile data. In particular, two sections of laser profile data, one section being multi-year ice and one section being young ice, have been analyzed.

The power spectra for each segment of the profile was divided into a plot emphasizing the short period spacing and one emphasizing the long period spacing. This was accomplished by appropriate block averaging of the filtered time series data prior to processing. The power spectra plots for each segment have then been superimposed in Figure 1, thus permitting direct comparison. The relative amplitude of the power spectrum has been plotted versus lag number. Lag number is a relative measure of spatial frequency. In both the short and

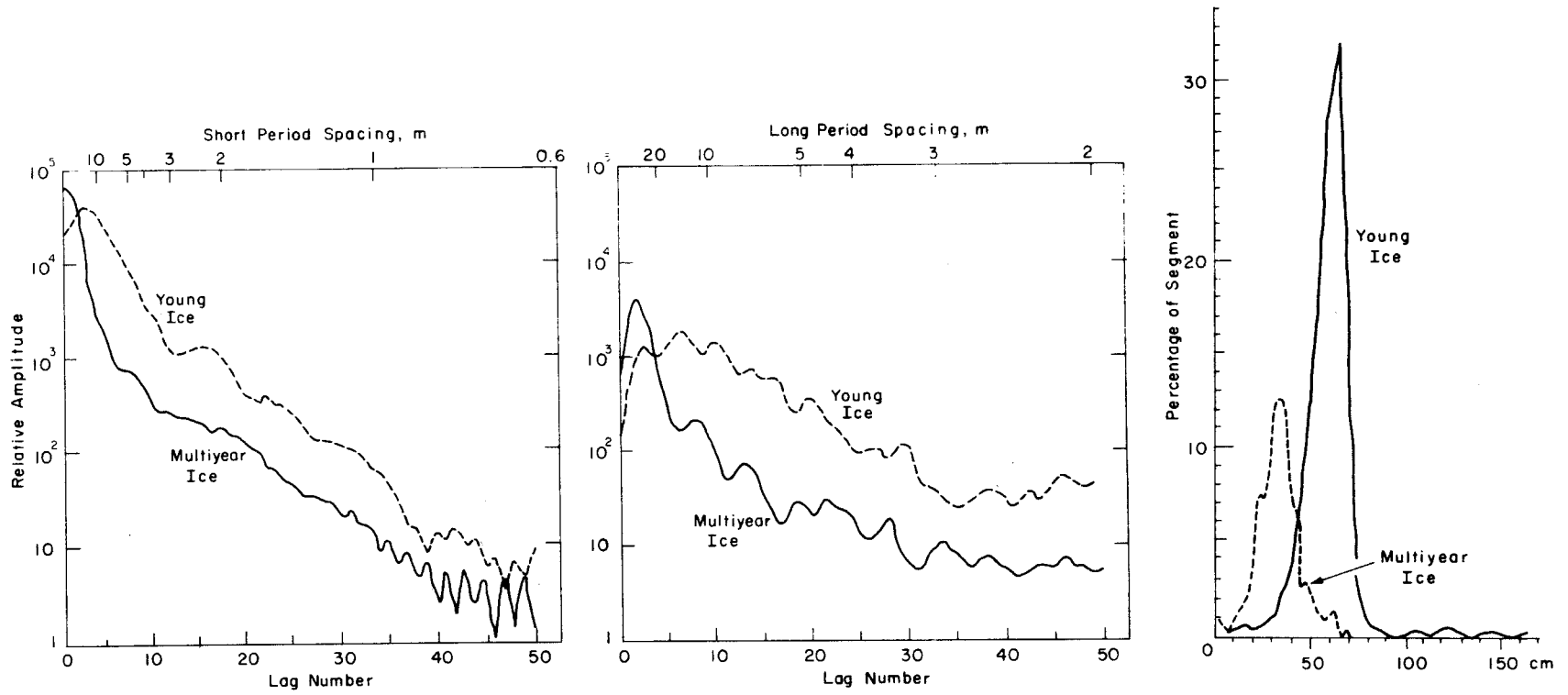


Figure 1. Power spectra and the frequency distribution of ice elevations from both young and multiyear ice as observed during the winter in the Central Polar Basin.

longer period plots, the most noticeable difference is the greater amplitude of all spatial periods shorter than 20 m in the young ice profile. Above 20 m, the multi-year ice has distinctly greater amplitude. Physically, this indicates that the older, more weathered ice is more undulating than the young ice, while the younger ice, although flatter, is rougher. This is, in fact, what is observed in the field.

The frequency distribution of amplitudes also provide insight into the character of the two ice masses. The multi-year ice has a broader, more even distribution of elevations, between the lowest and highest point on the profile, than the young ice, which is characterized by a sharper peak, centered at a greater elevation above the minimum point. The multi-year ice peaks at 35 cm while the young ice peaks at 60 cm.

There are apparent preferred ridge spacings on both profiles represented by power spectral peaks at periods of .75, 2, 3, 5, 7 and 13 meters. Statistically one would expect these spectral peaks to be real as the 95% confidence limits are between .7 and 1.6 times each estimate for the short period plot and .56 and 2.2 times each estimate for the long period plot.

The physical significance of such peaks, however, is not resolved and further analysis with this aim in mind is needed. The coupling of a dynamical model with power spectrum analysis might yield useful results in this area.

This type of analysis has also been applied to undersea sonar data and has been used to determine ridge lineation as well as significant ridge spacings for underice ridges.²

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